Applying the Kelly Criterion to Cashing In and Cashing Out

In this article I'm going to analyze how to maximize your expected growth (EG) when you have spots where you're given the option to cash out, and when you find spots where you want to cash in. We're going to be dealing with big numbers here, so while these spots are rare, they can make such a big difference in your overall bankroll that knowing how to navigate them correctly is very valuable. As I have in previous articles, I'll be applying the technique of Theoretical Kelly Optimization (TKO) to show you the mathematically correct answer. While these spots may seem to be very different from each other, I plan to show you how the optimization of your EG should be the foundation of your decisions in each case.

The Vanessa Kade Conundrum

Recently, a talented young poker pro named Vanessa Kade won a very large online poker tournament, with a 1st place prize of \$1.5 million. To make this accomplishment more remarkable, she did it without cutting any sort of deal to guarantee her a certain percentage of the prize money if she didn't win. So, the entire \$1.5 million was hers. While clearly an amazing feat on its own, her victory came only a few months after a very public beef with the infamous and obnoxious poker player/millionaire Dan Bilzerian, who said some very sexist things about her and claimed no one knew who she was. While there's been no indication that he recanted anything he said since then, let's turn this story into an interesting exercise to illustrate the expected utility of certain money vs. the chance at a bigger score.

Say Bilzerian had realized the error of his ways and, to try to make amends, offered Vanessa this deal: he would either give her \$1 million or flip a coin with her and give her \$X million if she wins. For what value of X should she be indifferent to taking either the guaranteed money or the flip? (Let's assume that the \$1.5 million she won represents her total wealth now, and she wants to decide based on the mathematical optimum).

As you may have guessed, the answer isn't simply based on the EV of the proposition. Despite her new-found wealth, the money at stake here is a significant fraction of that wealth and risking a large sum of money on a coin flip isn't something a smart poker player is likely to do without getting the proper odds. But how can we find the "proper" odds to help her cash in on his apology offer in an optimal way?

The trick is to frame the problem a little differently. Instead of a choice between sure money and a flip, an equivalent situation would be if he simply gave her \$1 million and then proposed to flip her for it back while offering odds of X-1. In both framings, she can walk away with \$1 million or take the flip and leave with either \$X million or nothing. Now we can set her certainty equivalent of the flip equal to \$1 million (given her new bankroll and the odds on a 50/50 proposition). What is her certainty equivalent? It's the guaranteed amount of money that's equally desirable to the gamble, as deftly described by Joseph Buchdahl in his book "*Squares and Sharps, Suckers and Sharks*" and by @PlusEVanalytics in his Pinnacle blog article "*The Real Kelly Criterion*." In short, it's when her EG = 0 for the flip. How can we calculate this amount without doing logarithms?

Well, recall that the simple Kelly formula calculates the fraction of your wealth to wager in order to maximize your EG, aka to maximize your marginal utility. In this case, however, we aren't trying to *maximize* her utility – we simply want it to be the same whether she takes the flip or not. This equivalence will happen when her amount at risk is twice the Kelly fraction, because with that much on the line her EG will be exactly 0, just like it would be if she kept the \$1 million he initially gave her. (Remember the simple Kelly criterion curve of EG vs. your bet size has its maximum at "*f*" and has EG = 0 for both no bet and "2*f*").

To apply it to this problem, recall that the simple Kelly formula is:

$$f^{\circ} = p - \frac{q}{b}$$

where *b* represents the fractional odds that we're trying to determine. But in this case, we don't want to use the optimal fraction. We want the fraction to be twice optimal, so we must multiply the right side by 2 like so and then solve for *b*:

$$f = 2 * (p - \frac{q}{b})$$

$$f = 2p - \frac{2q}{b}$$

$$2p - f = \frac{2q}{b}$$

$$b = \frac{2q}{2p - f}$$

Now we plug in $p = q = 0.5$ for a 50/50 coin flip and $f = 0.4$ (because $1 million is now 40% of her bankroll):

$$b = \frac{2 * 0.5}{2 * 0.5 - 0.4}$$

$$b = \frac{1}{0.6}$$

$$b = \frac{5}{3} = 1.667$$

Now to find X, recall that in our new framing the odds Bilzerian offered her were X-1, so:

$$X = b + 1 = 2.667$$

and the amount that should make her indifferent to taking the flip is $2.667 million. This makes the EV of the flip $1.333 million, meaning that she may be correct to pass up a reasonable edge on the flip in order to reduce the risk.

For a billionaire like Jeff Bezos, *f* is very small and *b* approaches 1. For him, anything above 1:1 odds is a bet worth taking and X ≈ 2. For most of us, *f* is very large. For example, if you have $10,000 to your name then *f* = 0.99, and X = 101. So you'd be wise to hold out for over $100 million on the flip rather than take the cool million. It seems your frame of reference shifts dramatically depending on your own personal situation. Of course, real life isn't just about the math and sometimes there are reasons to deviate from a mathematically optimal answer. On the other hand, knowing what that answer is can provide a very important reference point when making decisions about cashing in on potentially life changing money.

The Swap Equivalent

Suppose you made a futures bet on Baylor to win the NCAA basketball tournament, and after a thrilling ride to the Final 4 they make it to the championship game. Whether or not you had +EV when you made the bet, you have quite a bit of EV now. You stand to win a large fraction of your overall bankroll, but there's still a lot of risk involved. You consider hedging with a bet on Gonzaga, but that involves putting up a large fraction of your bankroll on that side, and maybe you don't have access to a sharp book so you may not get a very good line. You also might have the option to cash out your ticket, i.e., to sell it back to the regulated book where you placed the bet or to a 3rd party – and for a lot more than you bet in the first place. So many options… what should you do?

If you're considering cashing out, how can you tell if you're getting a good deal or getting ripped off? Truth is, it depends on how likely you are to cash that ticket if you keep it. You can calculate your EV quite simply if you have a good estimate of that likelihood, since:

$$EV = pW$$

Where:

p = probability that the existing bet wins

W = actual payout returned if existing bet wins

If you can cash out for that value or more, it should be intuitively obvious that it's correct to do so rather than letting it ride (aka standing pat with your existing bet). Guaranteed money is always better than the same EV with some risk. But what if you have to give up some of your EV? How much of an "insurance" premium should you pay? Does it depend on your personal risk tolerance? Sure, if you enjoy the sweat of trying to hit a big score then you may hold out for an amount that's very close to your EV. If you're more of a nit, you may be willing to take a lot less. But there is a mathematical optimum, and by knowing it, you can try to calibrate your emotional response to better align with the strategy that will make you the most money in the long run.

So how can we calculate this optimum? First, recall that the method that grows your money the quickest is maximizing the expected value of the logarithm of your overall bankroll. This is what TKO seeks to do in situations where the simple Kelly criterion equation doesn't apply. Through this lens, we want to see how much "certain money" is equivalent to the ticket that we hold. But, even more useful in this case, would be to define a "swap equivalent" that we calculate as the percentage of your ticket's EV where you'd be indifferent to cashing it out or letting it ride. To do that, we need to construct the formulas for the logarithm of your wealth in each case and then set them equal to each other.

When letting it ride, you increase your bankroll by *w* when you win, but it's unchanged when you lose since you've already bought the ticket:

$$E = p * \log(1 + w) + (1 - p) * \log(1) = p * \log(1 + w)$$

If you cash out for the "swap equivalent" instead, then your bankroll is increased by that percentage times your EV, so we get:

$$E = \log(1 + spw)$$

Where:

s = swap equivalent (percentage of your EV where you'd be indifferent to cashing out or not)

p = probability that the existing bet wins

w = payout returned if existing bet wins (as a fraction of your bankroll)

To find the swap equivalent, we must set these two formulas equal to each other and solve for *s* like so:

$$E = p * \log(1 + w) = \log(1 + spw)$$

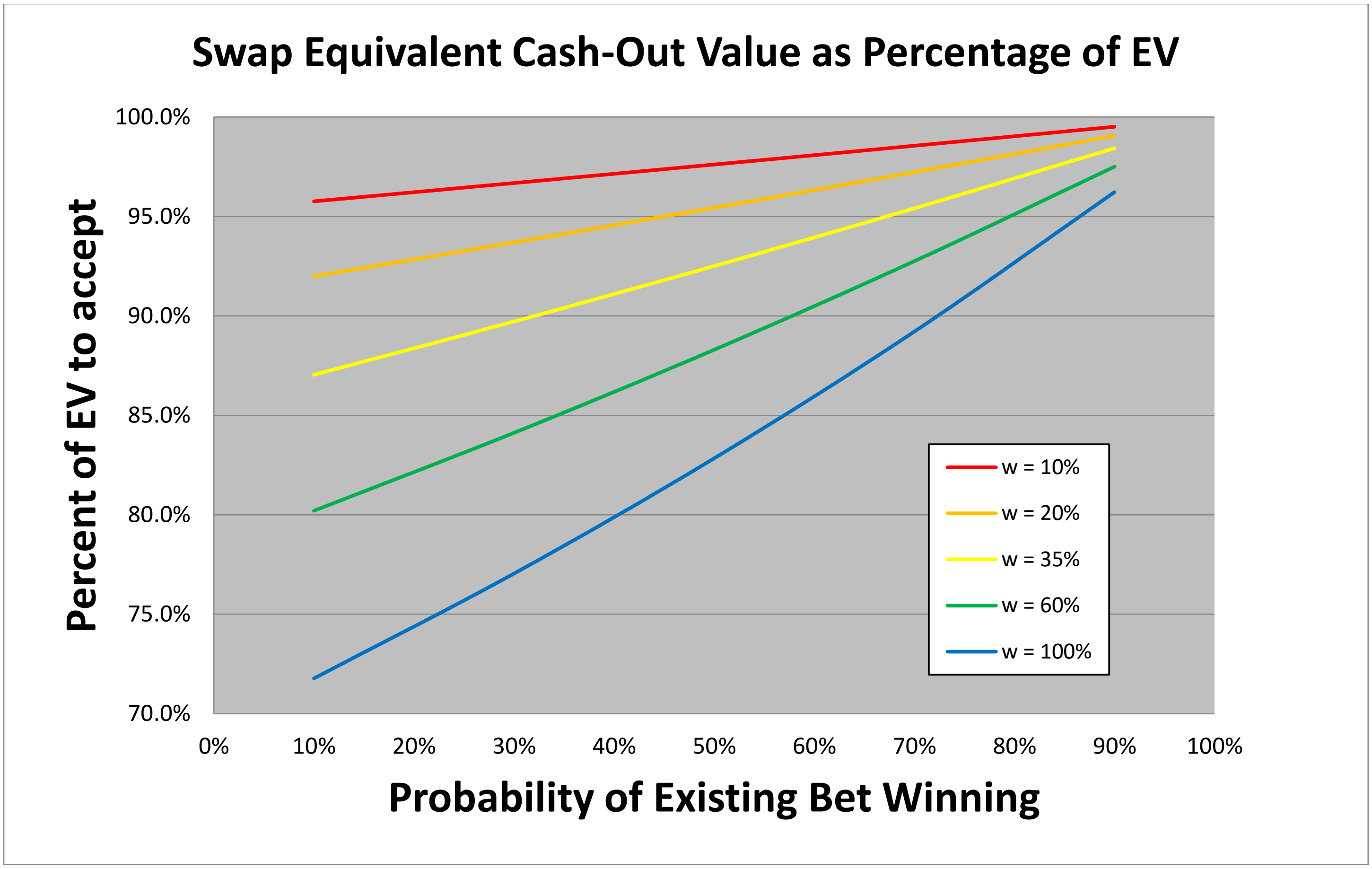
$$\log(1 + w)^p = \log(1 + spw)$$

$$(1 + w)^p = 1 + spw$$

$$(1 + w)^p - 1 = spw$$

$$s = \frac{(1 + w)^p - 1}{pw}$$

Notice that this answer is a little more complex than the formulas for hedging and doubling down, but at least we didn't have to do any calculus this time! Also, notice that the answer doesn't depend on how much you bet initially. That money is in the book's account and shouldn't affect your decision about what to do now. In the chart below, I've plotted a family of curves that represent different values for *w* (your payout in percent). For each one, I compare the probability of your existing ticket winning to the minimum value of *s* that you should accept. As common sense will probably tell you, the more you have at risk, the lower the swap equivalent can be to make cashing out worthwhile. After all, the amount of EV you give up is like an insurance premium, and you don't need insurance on your new toaster as much as you do for your car or your house. On the other hand, when you're almost certain to cash your ticket you really want to get almost full value for your cash-out.



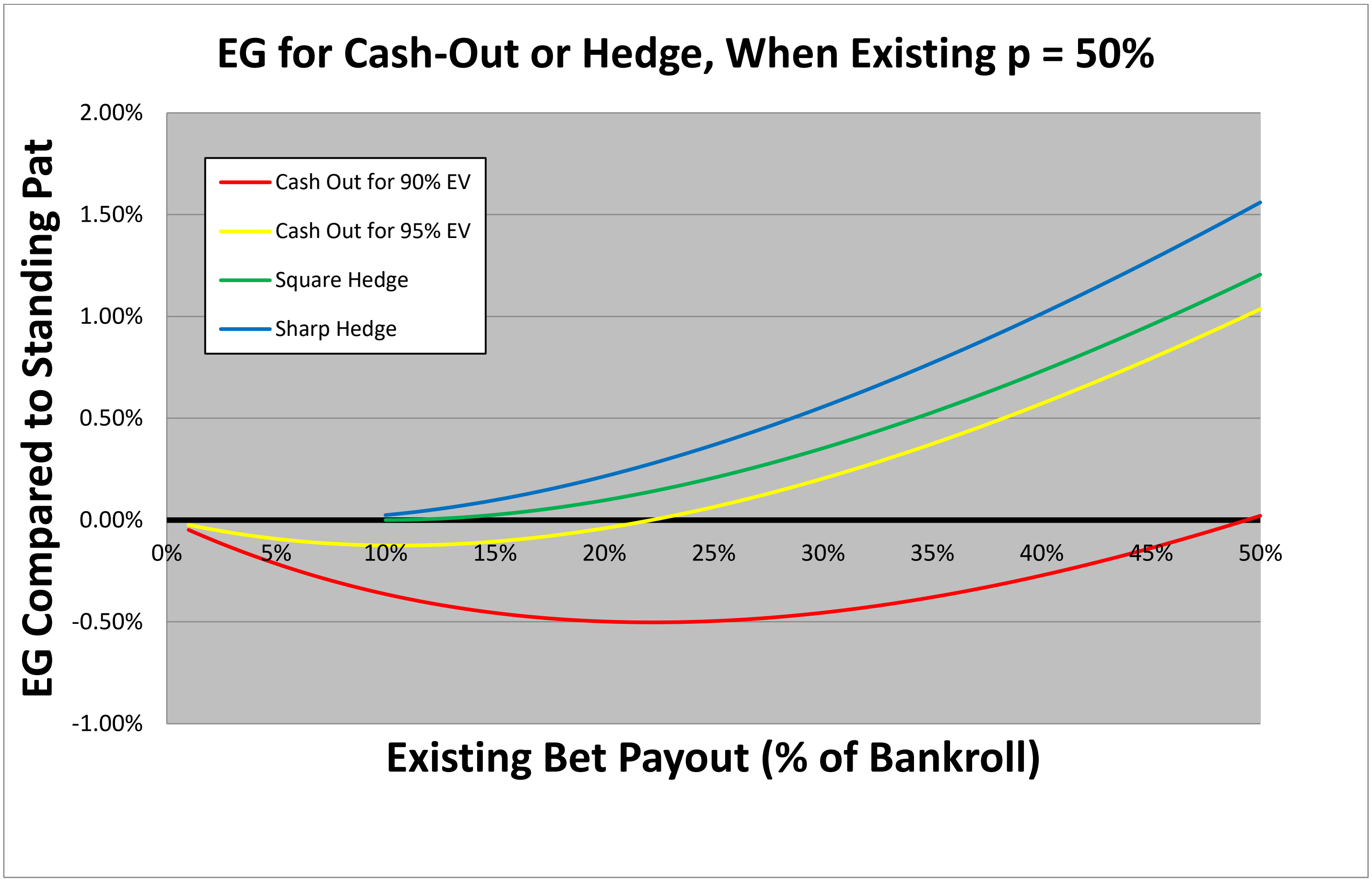
When *w* is 10% or less, you should hold out for almost full value on a cash-out. When your potential payout is larger, you need less and less value to make cashing out correct. In fact, when you have the possibility of doubling up your bankroll (*w* = 100%), then your swap equivalent can range anywhere from 96% to 72% depending on how likely that is.

All in, All out, or Just a Bit?

Suppose you have all of your options available in your quest to maximize your EG. If you're lucky enough to have access to a sharp book that gives reasonable lines and very high limits, you could hedge on Gonzaga there. If you're not, maybe you have to get down a few smaller hedges on soft books and take whatever line they give you (typically with a vig of 4-5%) A cash-out will often cost you a 10% fee or even more, but sometimes a regulated US book will offer you a better deal than that. So how much of a difference does the vig really make when deciding which option is best? To find out, I subtracted out the EG of standing pat and then plotted the following charts to compare your EG for these four spots:

1. Cashing out for 90% of your EV
2. Cashing out for 95% of your EV
3. Optimally hedging at a sharp book (with ~2% overall vig)
4. Optimally hedging with a square line at a soft book (with a ~4.5% overall vig)

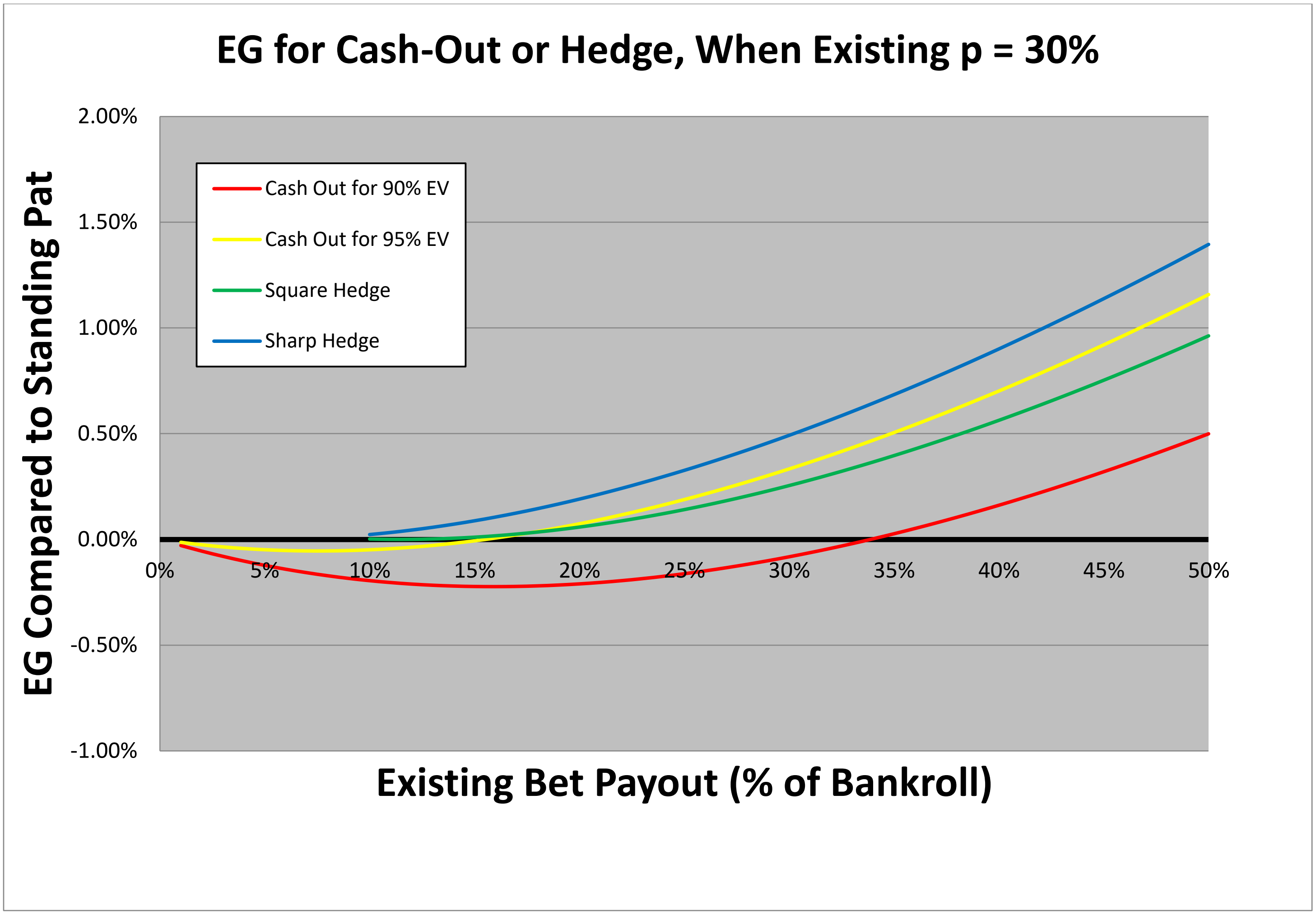
In this case, I plotted the curves versus your potential payout, and only took it up to an increase of 50% of your bankroll so it's easier to see the detail for smaller payouts. Say Gonzaga's best player was injured in the semi-finals, and now it's a 50/50 proposition that you'll strike gold with your Baylor future. You have a lot of EV on the line and hedging will almost always be your best play, as you can see from the chart below:



With this high a chance of winning, payouts of less than 50% of your roll don't favor cashing out for 90% of your EV instead of standing pat, because your increase in EG would be negative (in these charts, standing pat is where EG = 0% since I subtracted out its EG from the data). If your swap equivalent is 95%, then payouts of 25% or more are worth cashing out, but for payouts of any size you do better by hedging than by cashing out. As you may have guessed, the difference between the strategies would be even greater if Baylor were the favorite. Why should that be? It's because a cash-out is all-or-nothing but when you hedge optimally you only "lock up" a portion of your holding and let the rest ride, paying less in vig and striking just the right equilibrium between risk and reward. In other words, it answers the question of should you sell or hold with another question: why not both?

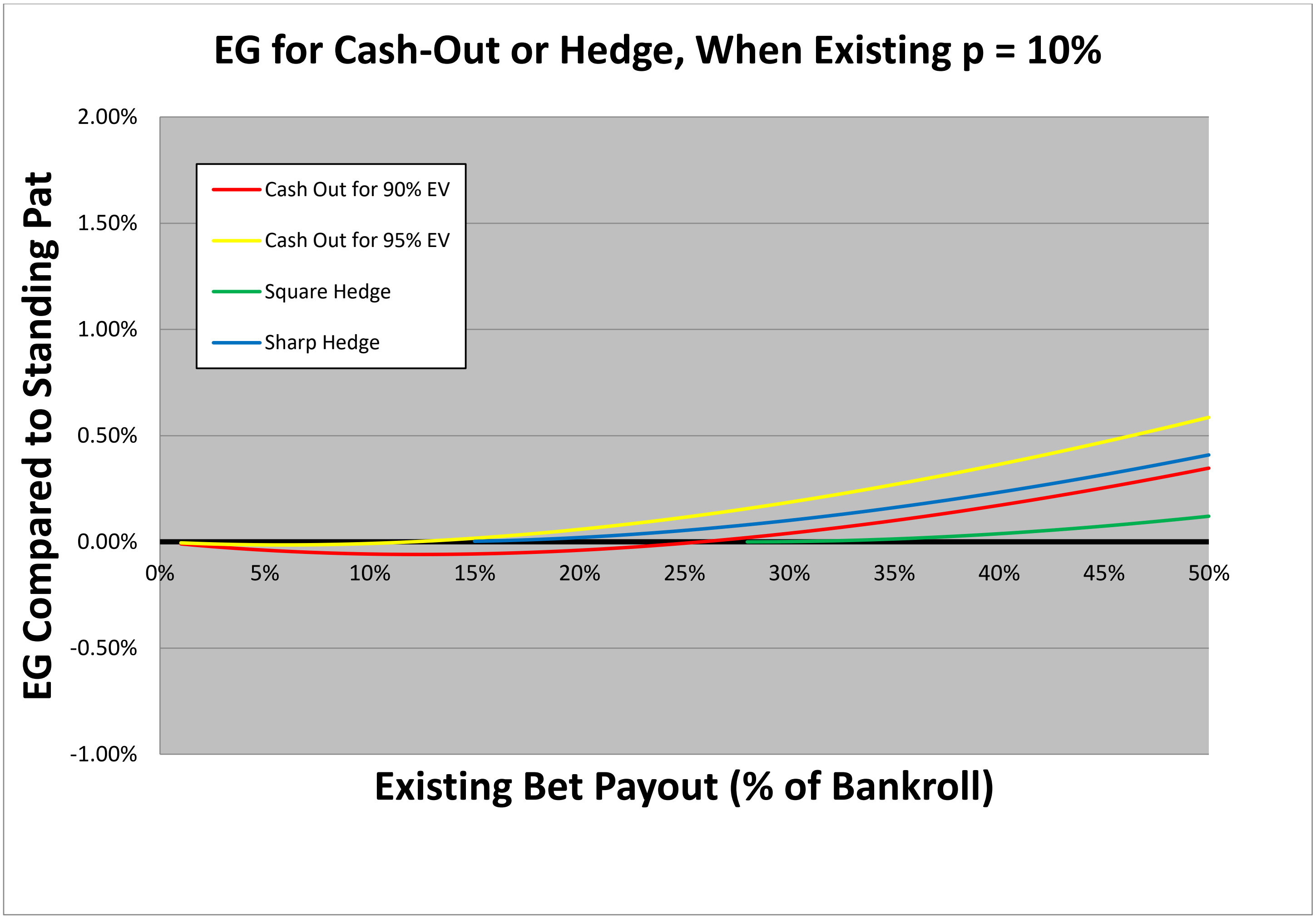
While our TKO evaluation says you should be hedging in this spot, it may be difficult to do in practice. For a square hedge you'd have to get down 27% of your bankroll and for a sharp hedge you'd have to bet almost 32% (based on a sharp line of Gonzaga -105 and a square line of Gonzaga -110). For most bettors, that's a large sum of cash and you'd probably need to plan accordingly before Baylor even makes it to the finals.

If the Bears have just a 30% chance of winning, the differences between your options become a little muddier like in the chart below:



In this case, only payouts of less than 35% of your roll aren't worth cashing out for a swap equivalent of 90%, since your increase in EG vs. holding is < 0%. Any ticket with over 10% of your bankroll at stake is worth hedging against, however for payouts of 20% and above you actually do a bit better by cashing out for 95% than you do by hedging with a square line of -260/+210 (approx. 4.5% vig). What's up with that? The problem you run into when hedging on Gonzaga is that you have to bet on a 70/30 favorite so the simple Kelly fraction is more negative (i.e., -8%) than when betting at even odds with the same vig. This means that, when added to your potential payout, the simple Kelly fraction significantly reduces the value of your hedge. It may be unusual to find a swap equivalent of 95%, but if you can it's certainly worth a look!

Say Baylor had a Covid problem instead, and they wound up as a true 9-1 dog against Gonzaga in the finals. When you're dealing with a serious long shot like that (i.e., with a 10% chance of winning), you may need a magnifying glass to see which option is best for w < 20%. But for larger payouts, the simple Kelly fraction of -20% on Gonzaga skews our results even more like in this chart:



For payouts of less than 20% of your roll, a sharp hedge and a cash-out for 95% have about the same EG, a 90% cash-out is slightly worse, and a square hedge can't even be considered. For payouts above 25%, a swap equivalent of 95% comes out on top because its fee is the same regardless of whether you're on the favorite or the dog. Even cashing out for 90% looks good compared to hedging, especially since the optimal hedge would be to wager 34% of your bankroll at a sharp book and 20% when making a square hedge (based on a sharp line of Gonzaga -1000 and a square line of Gonzaga -1150). This is the kind of spot where cashing out will usually be your best play, despite your intuition that it's "wrong" to give up so much EV.

It's All Relative

What do these spots have in common that lets us evaluate them in a similar way? Just like when comparing doubling down to hedging, in some cases we have +EV bets that are actually -EG, and in others we have -EV bets that are +EG. This apparent paradox happens because gains and losses are not the same for all bettors. How good or bad they are depends on your frame of reference, i.e., how large they are relative to what you already have and what you already stand to gain. It follows that if there's no "right side" only the "right number," then there really is no "right number" only "the right number *for you*." This is the betting theory of relativity, and it doesn't just apply to spots that are technically called "bets."