Applying the Kelly Criterion to an event with existing bets, part 3

The title of this article is a lie. It should actually be called "Staking strategies that may require you to place a -EV bet." While it is a follow up to my two previous articles on the subject, I won't be applying the Kelly criterion to existing bets or analyzing what to do with bets you've already made. Instead, I'm going to take a deep dive into applying the Kelly criterion to making a strategic plan for staking all your bets before you make them. We started to do that in Part 2 with the Neutral Hedge Gambit, but using what we have learned about hedging and the value of risk reduction, we can analyze the difference between value betting (i.e., betting one side of a market when you estimate that it's +EV) and arbitrage betting (i.e., betting both or all sides of a market when the respective odds are high enough to guarantee a profit). By figuring out that difference, you'll see that your best strategy will often require you to place -EV bets.

Recall that the simple Kelly Criterion equation for independent bets is usually written as either:

$$f^{\circ} = p - \frac{q}{b} \;\; or \;\; f^{\circ} = \frac{bp - q}{b}$$

Where:

f = fraction of bankroll to bet

b = net fractional odds of the bet (for American odds +200 → +200/100 = 2, -200 → -100/-200 = 0.5)

p = probability that the bet wins

q = probability that the bet loses, or 1 - *p*

You should also recall the rules and assumptions that make this fraction the optimal amount to stake on a bet:

1. You're offered a single kind of wager at specific odds and with a well-defined probability of the outcomes.
2. These odds and probabilities don't change, either before the trial that you have bet on or in between trials.
3. You can have as many goes as you want, but they must be in sequence (i.e., one at a time) and you must wager the same fraction each time.
4. After each trial, you must reevaluate the size of your bankroll and stake the prescribed fraction of that new bankroll on the next trial.

Does this sound like a scenario that a typical sports bettor encounters very much? I didn't think so. This is the number one reason why the simple Kelly formula has failed so many bettors and prompted one long-time pro to say that betting full Kelly will "make you go broke." It's a great tool, but you can't drive a nail with a circular saw and you can't use this simple Kelly formula to optimize your expected growth (EG) in more complex situations. You can, however, form a strategy that seeks to optimize your EG in every situation by using the underlying math of the Kelly Criterion. Theoretical Kelly Optimization (TKO) does this by finding the equilibrium between risk and reward that maximizes your median bankroll, and all it takes is three simple numbers: the odds offered on a bet, the true odds of that bet, and the size of your roll.

Risk vs. Reward

Let's say you're a sports bettor. A sharp one. Say you're line shopping and see a great spot for the first half money line on a basketball game with the following odds:

Soft book odds: Team A +400/Team B -550

Sharp book odds: Team A +315/Team B -350

The limit to wager on Team A at the soft book is $100 (making the limit to win $400), and the limit to win on Team B is $400 (making the limit to wager $2200). The limits at the sharp book are far more than you have to worry about, because your total bankroll is $2500, with $1000 in the sharp book, $500 in the soft one, and the other $1000 in other outs.

What's your play? Being a sharp bettor, you notice that the two sets of odds offer an arbitrage opportunity by betting on Team A at the soft book and Team B at the sharp one. You calculate that by betting the $100 limit on Team A, you can bet $388 (or 15.5% of your bankroll) on Team B at the sharp book, and reap a guaranteed profit of $11 with a balanced arb. Not bad! On the other hand, you realize that the sharp book has a ~2% vig built into their lines, and if we infer a vig-free line of ±331 (using a probit scale), then it predicts that Team A will win 23.2% of the time. It feels wrong to give back some of your EV to the sharp book in order to lock up a profit, so you want to calculate the right stake for a value bet instead.

If you assume that the sharp book is right, then by complete coincidence your full Kelly stake size would be $100 to bet on Team A. Here is the math:

$$f^{\circ} = \frac{bp - q}{b}$$

$$f^{\circ} = \frac{4 * 0.232 - 0.768}{4}$$

$$f^{\circ} = \frac{0.16}{4}$$

$$f^{\circ} = 0.04 \; or \; 4.0\%$$

$$4.0\% \; of \; \$2500 = \$100$$

And for completeness' sake, your full Kelly stake on Team B would be:

$$f^{\circ} = \frac{bp - q}{b}$$

$$f^{\circ} = \frac{0.286 * 0.768 - 0.232}{0.286}$$

$$f^{\circ} = \frac{-0.0124}{0.286}$$

$$f^{\circ} = -0.043 \; or \; -4.3\%$$

So is a full Kelly bet on Team A the correct play? You used a market-determined win percentage to estimate your hefty 16% edge, so you can consider it a square edge and have much more confidence that a full Kelly stake is optimal. Then again, sharp books can be wrong too and if you bet half Kelly instead, then you don't give up much EG, but your amount at risk is cut in half. Take a minute to work out your answer before reading on…

Why not both?

Maybe a compromise is best. But how do you figure out the optimal compromise? And will it be better than any of the options we already considered? We have to start answering these questions by defining the nature of a balanced arbitrage, where you have no money at risk and make an equal profit regardless of who wins. There are two ways this can happen: when one side has a very large edge and the other has slightly negative one, or when both sides have a slightly positive edge. Obviously, this example falls into the first category, and that's the one we want to analyze to compare staking strategies that may require you to place a -EV bet.

Since it's a two-step process to pull off an arb, now suppose you bet on Team A first, then fill out your slip to bet on Team B. Before you hit "place bet," think about how else you could have gotten to an equivalent spot. Maybe your bet on Team A was actually made months ago as a "Tampa Bay to win the Super Bowl" future at 19-1 (like in Part 1 of this series), but Brady was injured in the NFC championship and now they're a huge underdog in the Big Game. Either way you got to this spot, your bet on Team A pays out 20% of your bankroll and now you have to decide how much is optimal to bet on the opposite side. So, either way you just use the opposite-side Kelly formula to find the answer like this:

$$f^{\circ} = p - \frac{q}{b} + pw$$

$$f^{\circ} = 0.768 - \frac{0.232}{0.286} + (0.768)(0.20)$$

$$f^{\circ} = 0.768 - 0.811 + 0.154$$

$$f^{\circ} = 0.11 \; or \; 11.0\%$$

Where:

p = 76.8%

q = 23.2%

b = 0.286 (i.e., -350 in American odds)

w = 20%

So that's it. That's the answer. But what's really going on under the hood? Recall that to make a balanced arb, your bet on Team B has to be 15.5% of your bankroll, and the optimal bet to make on Team B from a Kelly perspective is about -4.5%. So, why not both? Because your bet on Team A is already maxed out, the optimal play is to reduce your bet size on Team B by adding the negative Kelly fraction that we calculated above:

$$15.5\% + (-4.3\%) \approx 11.0\%$$

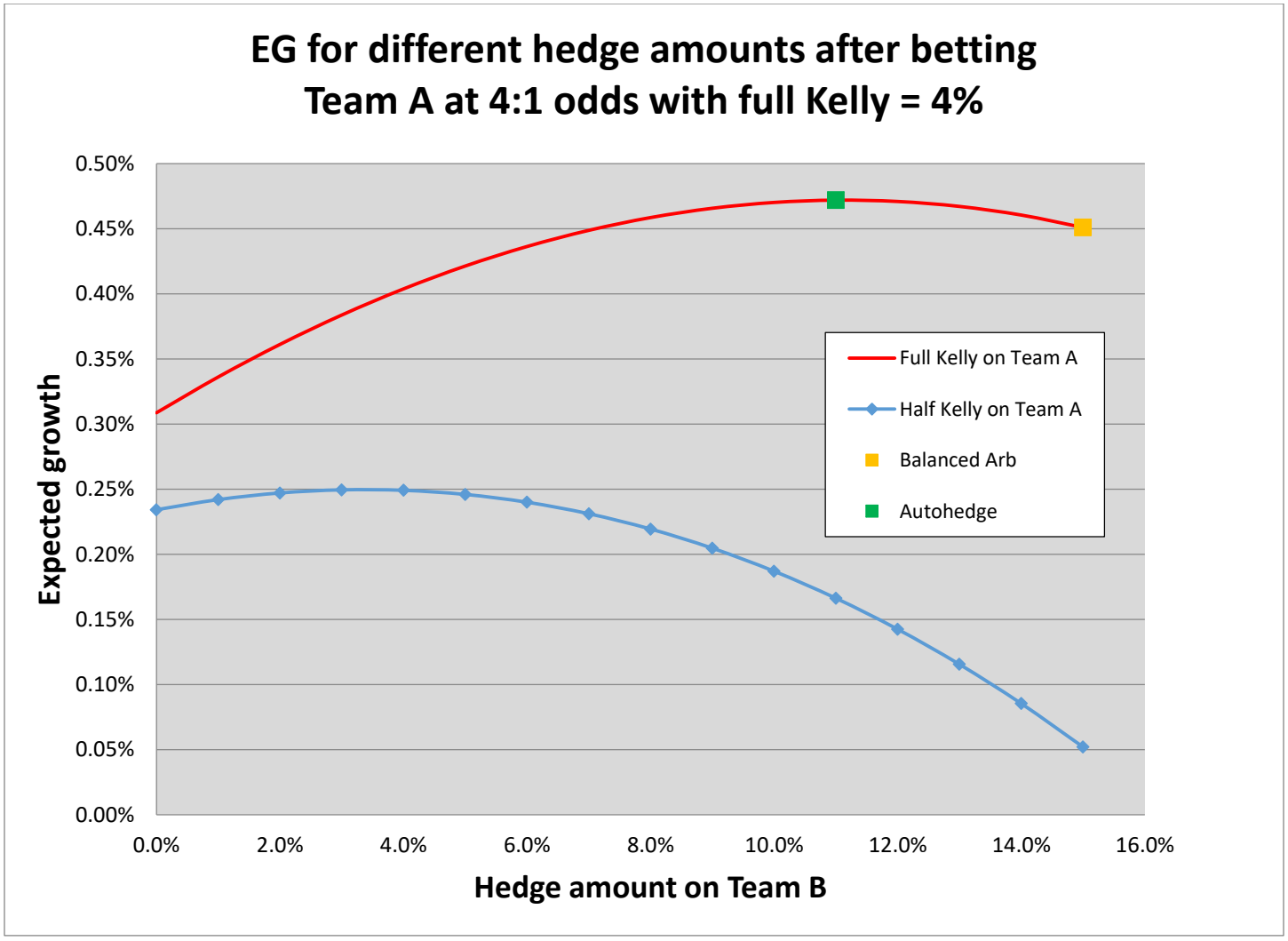
(If your bet on Team B was limited instead, then you'd optimize your EG by making a balanced arb and then betting an extra 4% on Team A.) This method is like a strange hybrid vehicle where you weld a value bet onto your arb while betting as much as you can on the combination. So, since the idea is to automatically hedge after making your initial bet, I welded the words "automatic" and "hedge" together to call it an:

$$Auto(matic) + hedge = Autohedge$$

Pics or it doesn't happen

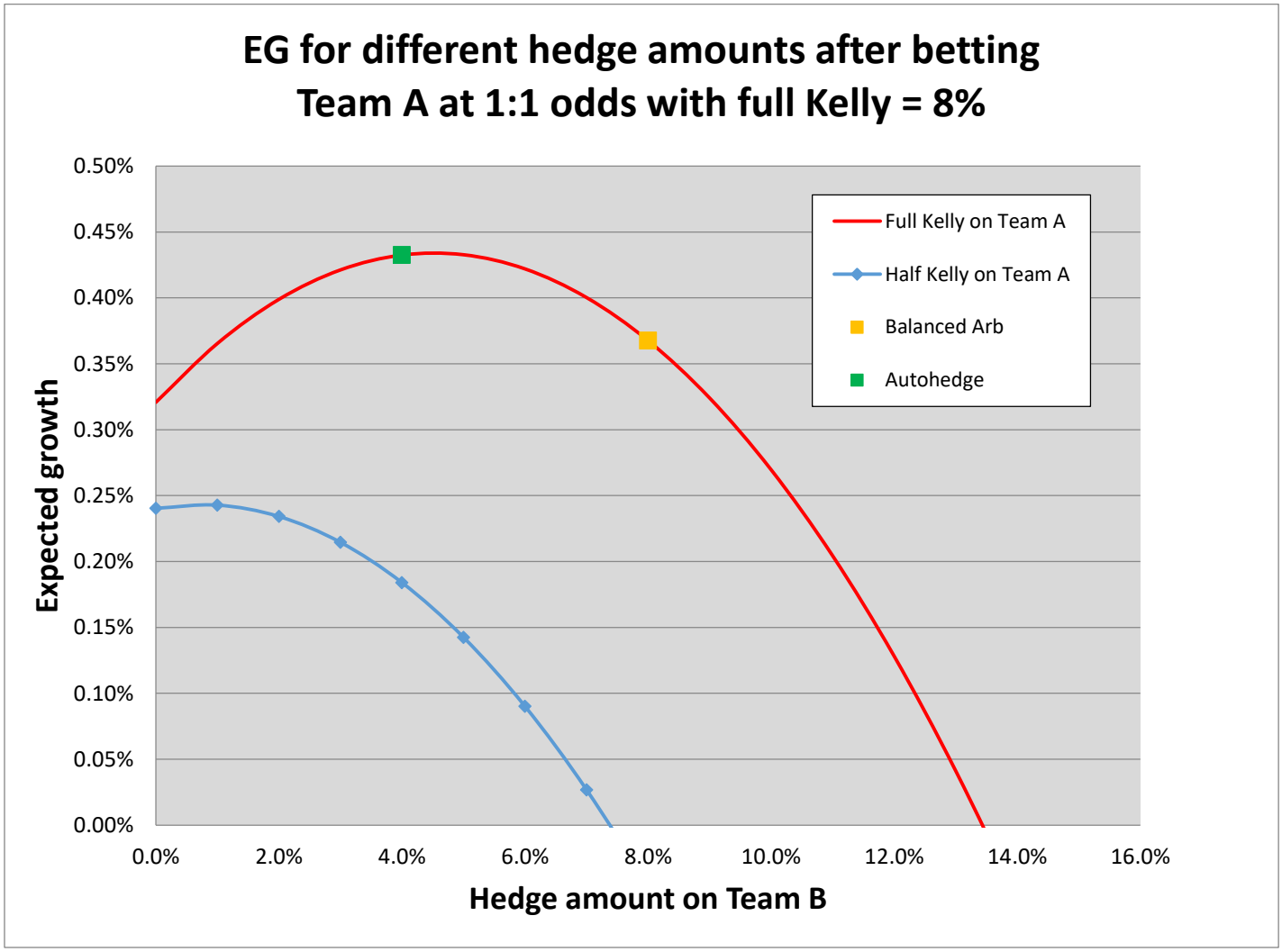
To illustrate how your EG changes with hedge size, I plotted two curves in the chart below. The red curve represents your EG when using full Kelly staking, with no hedging at "Hedge amount on Team B" = 0.0% and increasing up to a hedge of 15%. I put a green marker at a hedge size of 11% to show you the EG for an autohedge, and a gold marker at 15% to roughly indicate a balanced arb. The blue curve happens when you start by staking on Team A with a half Kelly size (i.e., 2% of your bankroll), and it's pretty obvious that it doesn't benefit you much to hedge in that case, since the blue curve is almost at its peak when hedge = 0%. Of course, it's also pretty obvious that you're costing yourself a lot of money, because your bankroll will grow twice as fast if you autohedge instead. You're also costing yourself money if you bet full Kelly without hedging at all, because the full Kelly curve is lowest at 0% hedge.

Since the full Kelly curve is so flat near the top for 4:1 odds on Team A, your increase in EG by autohedging instead of arbing isn't dramatic. But what happens if we shorten the odds on Team A to even money?

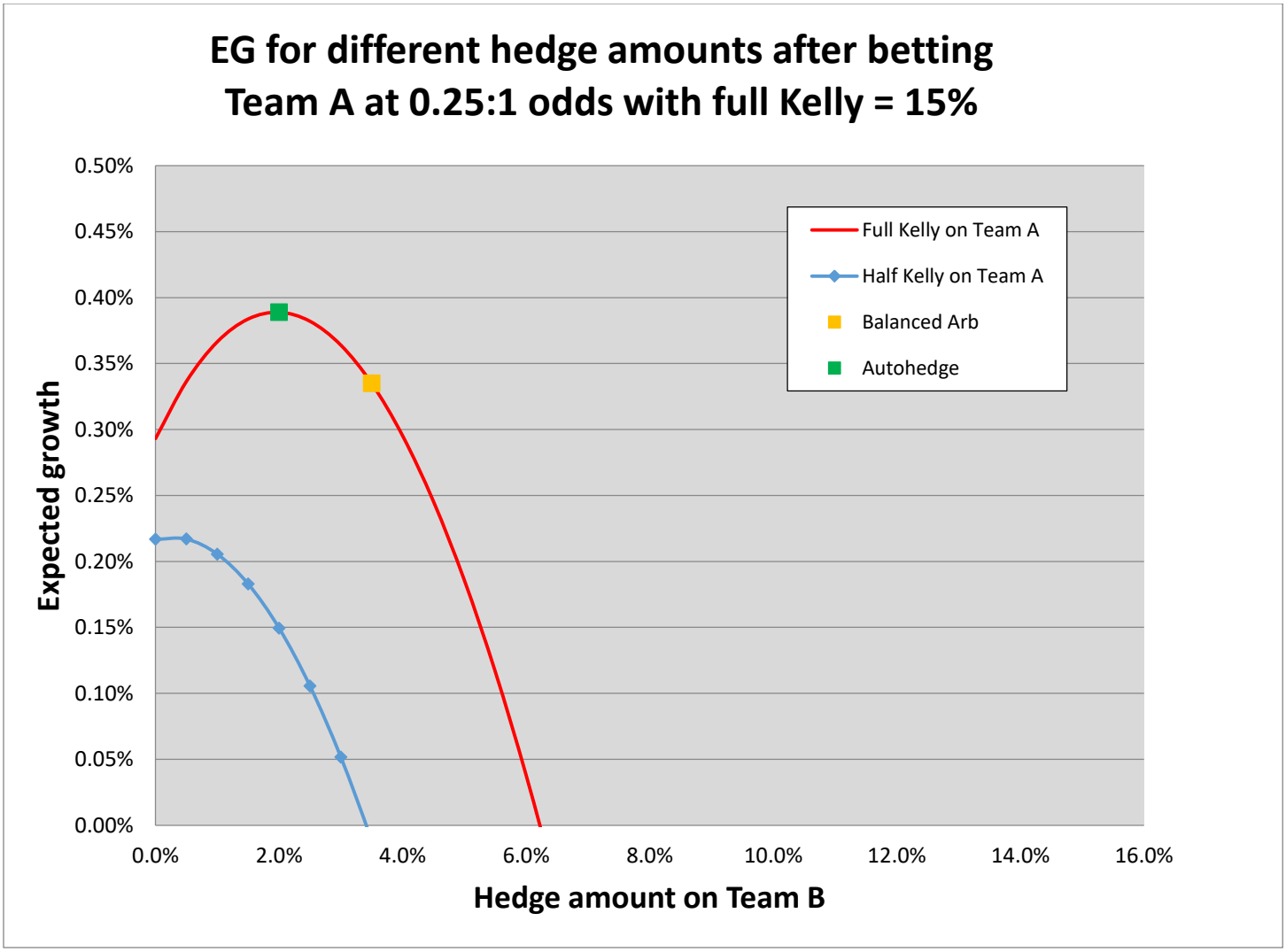


The next chart shows the even money case, and for those odds your EG is significantly better if you autohedge than if you strictly arb or make either full or half Kelly value bets. In this example your edge is down to 8%, but that's much more realistic for the even money case since it represents Team A/Team B lines of +100/-120 at the soft book and -

120/+110 at the sharp one. With these odds, an autohedge has almost 20% greater EG than a balanced arb, and is still clearly superior to either of the simple Kelly bet sizes with 0% hedging.

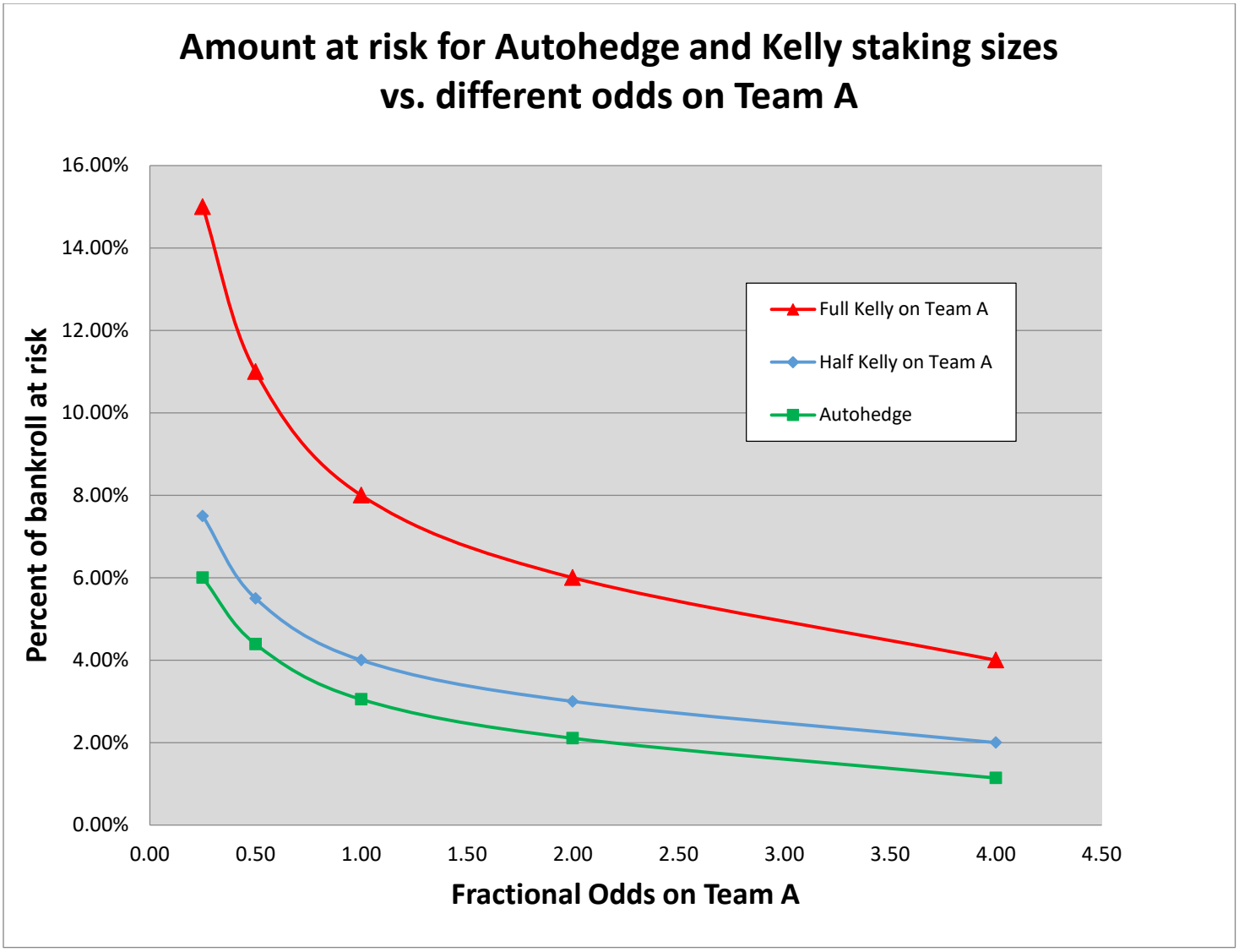


When Team A is a big favorite, you need to get down much more on them than you do on Team B, but there are still plenty of soft books that will let you bet 15% of your roll on -400 odds, since your amount to win is less than 4%. With -400 (aka 0.25-to-1) odds on Team A and +450 on Team B as in the chart below, an autohedge bet of 2% on Team B strikes the optimal balance and yields a much higher growth rate than a full or half Kelly value bet.

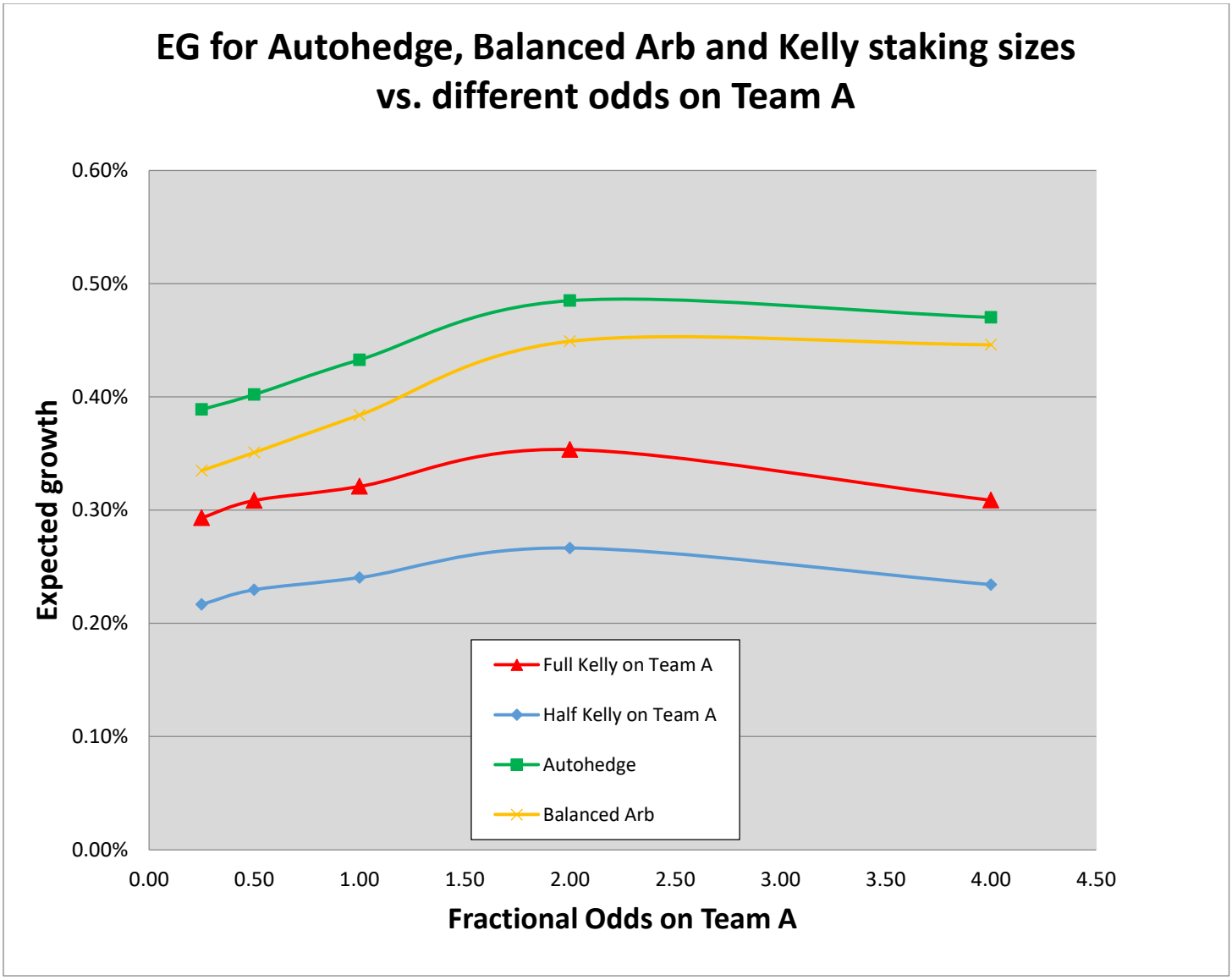


Since you'd only place a 2% bet to win 9% on Team B when you autohedge, you might have a bit more money at risk than you're comfortable with. Yet, when these rare opportunities come up, you will end up risking less than you do for even a half Kelly value bet. Think about it: if full Kelly staking on Team A is 15%, then you put 7.5% of your bankroll at risk by staking at half Kelly. With an autohedge, you'd wager 15% on Team A to win 3.75% and 2% on Team B to win 9% at +450 odds. This way, the most you can lose is 15% - 9% = 6%. In fact, regardless of which team is the favorite, you always risk less of your bankroll by autohedging than you do with half Kelly staking.

As you can see in the chart below, your amount at risk is roughly equivalent to 1/3 Kelly staking, so you can avoid the gut wrenching swings that come along with full Kelly sizing on these golden opportunities:



But is it really that much better to bet the opposite side while paying a 2% vig (and giving up some of your massive EV) in order to reduce your risk? Is autohedging better than arbing which has zero risk? Is it easy enough for most bettors to figure out how much to optimally hedge? As you can see from the summary chart below, the answer is yes, yes, yes! Yes, you can gain much more EG by arbing than by value betting at half Kelly or even full Kelly, and you gain even more EG by autohedging to hit that sweet spot in the middle. Yes, you get slightly more EG with an autohedge than with a balanced arb, and by using the TKO solution you don't have to bet very precise amounts or worry about the line changing before you can complete your trade. (Since you're betting the value side first, it's unlikely that the sharp book will move their line much when you go to make your hedge, but if they do you just need to adjust your hedge size slightly to account for the new line.) Yes, it's easy for everyone to determine the best hedge size, because all you have to do is use the basic opposite-side Kelly formula to calculate how much to bet on Team B (based on your estimate of the true win percentages and the amount you got down on Team A).



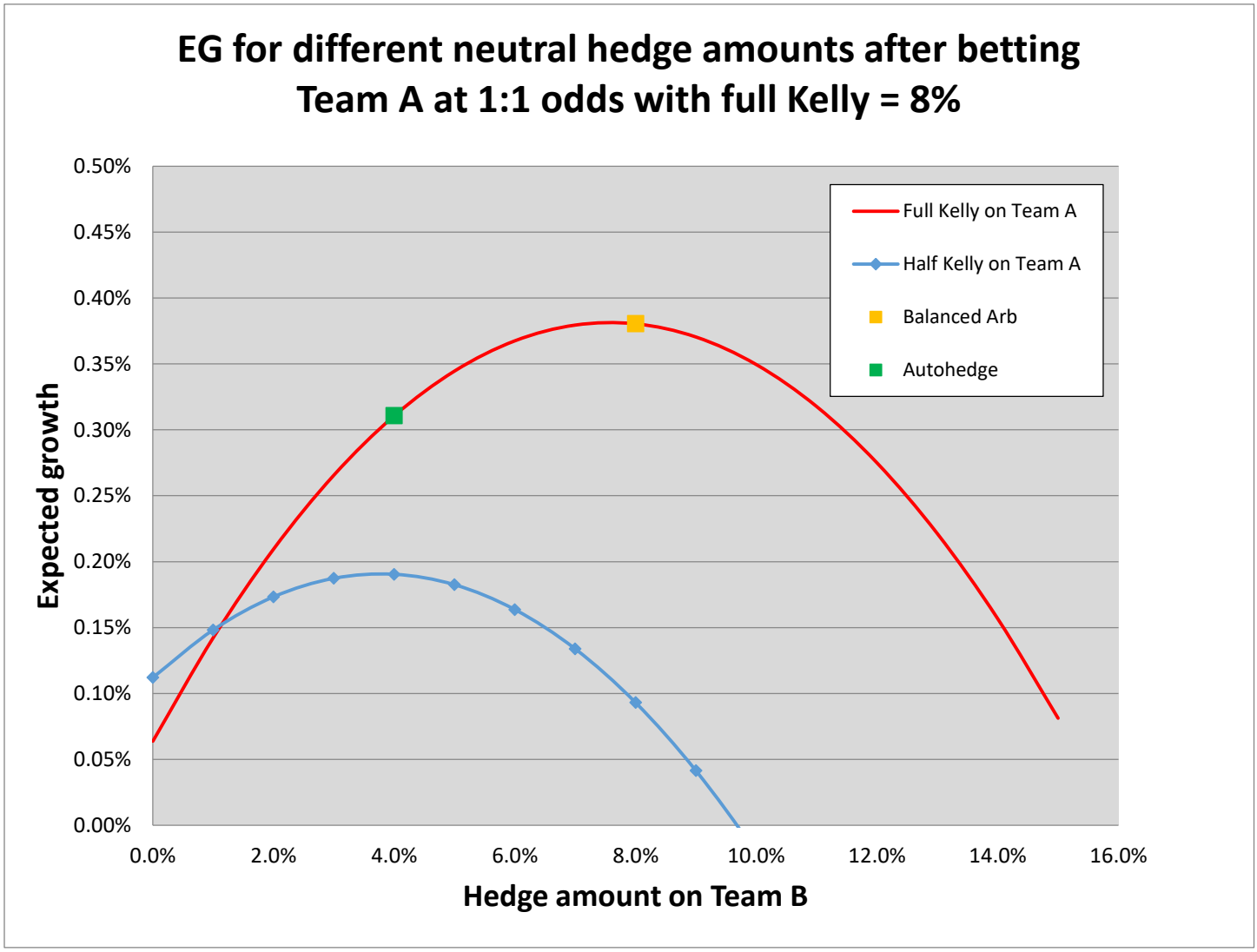
The best laid plans

But what if you're wrong? Or more accurately, what if the sharp book is wrong? They aren't infallible when setting their lines and, if you're counting on their handicapping of the game to calculate how much to hedge, you could be led astray. Thing is, they usually aren't wrong by much. A reasonable estimate of how wrong they're likely to be is that the fair line equals their line on Team B. If it's off by more than that, they'd be giving value on Team B which is something they're designed to avoid. If that's how it shakes out, then your hedge on Team B is actually neutral EV, but the value you thought you had on Team A is reduced. How do the different staking methods fare in this case?

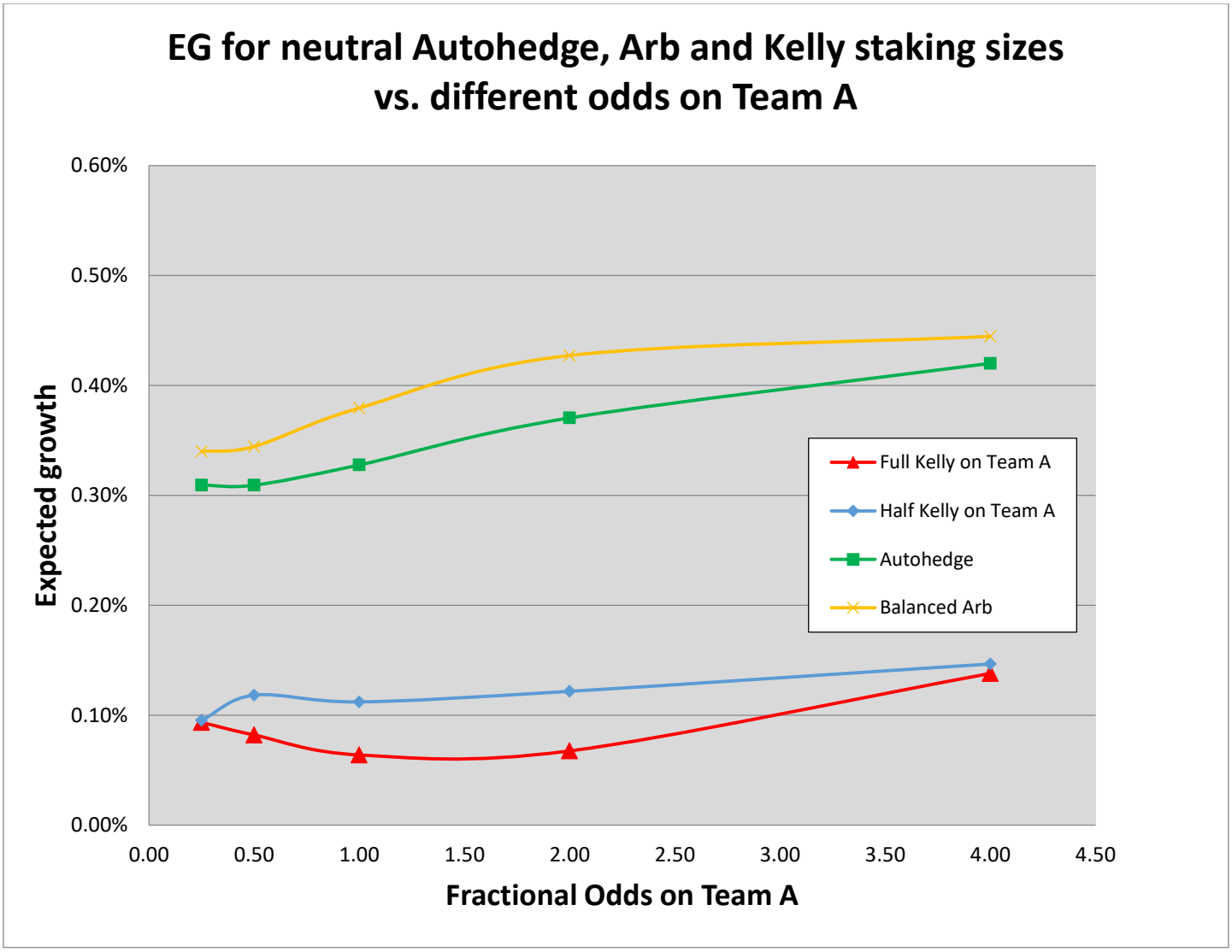
You can see a comparison of the EG for each one at even money odds on Team A in the chart below. Not surprisingly, betting the limit on Team A and arbing with Team B looks the same as before, because a balanced arb doesn't depend on the true win percentage on the game. It's only affected by the relative odds on both sides. That's the best play in this case, because the optimal hedge size when the simple Kelly fraction = 0% is just the amount that balances your payout for either side. An autohedge still does very well, but full Kelly is (almost) a disaster since you're seriously over bet. Half Kelly is better than full, but not nearly in the same league as either of the opposite-side strategies.

On the other hand, if the soft book limits you to a bet size of about half Kelly instead, what should your strategy be? Value bet, arb, or autohedge? By looking at the blue curve, you can see that hedging with a 4% bet on Team B optimizes

your EG when the sharp book line is off by this much. This strategy would completely eliminate your risk because you'd be freerolling on Team B (do you see why?) and who doesn't love a freeroll? However, if the sharp book is right, like in the first 1:1 odds chart, it would reduce your overall EG from 0.24% by value betting to 0.18% by autohedging your half Kelly stake.



The relative performance of each staking strategy holds true for all combinations of odds, as you can see in the chart below. Whether Team A is a favorite, dog, or pick-em, arbing or autohedging is much better than full Kelly or half Kelly value betting. A 2/3 Kelly sizing would bridge the gap a little, but how much you can lose with that scheme is twice as much as with an autohedge, so that strategy doesn't make much sense either. Quite simply, an autohedge makes it easy to win, and hard to lose.



If you have reason to doubt the sharpness of your sharp book, then in practice you may do best by splitting the difference between the amount you'd autohedge and the amount you'd bet for an arb (rounded to some reasonable dollar amount!). While this method still requires you to make a small -EV bet, after reading these three articles, you should be used to the concept that it's often best to make some -EV bets (just like it's best to bet a Kelly fraction of your bankroll on +EV bets, rather than going all-in to maximize your theoretical EV). Besides being more profitable than a balanced arb in almost all of these spots, autohedging has one final advantage. It's fun! It allows you to calculate the bet size that optimizes your EG while, at the same time, leaving you with some "skin in the game" to root for Team A. Gooooo Team A!!